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Engineering Notes

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Estimation of the Region of Attraction for State-Dependent Riccati Equation Controllers

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I. Introduction

NONLINEAR control has no general method for synthesis, which makes it more difficult for designers. Many of the current techniques require that the system satisfy some properties or require the construction of a suitable Lyapunov function. A somewhat recent nonlinear control technique is the so-called state-dependent Riccati equation (SDRE) control method [1]; its main property is that the nonlinear control design is structured as a linear time invariant (LTI)-like problem with state-dependent matrices. The resulting controller is then found by solving a nonlinear Riccati equation online.

One of the main advantages of this method is that it is systematic, and it can be applied to a large class of nonlinear systems. Unfortunately the stability properties of SDRE-controlled systems can be guaranteed only in a small neighborhood of the origin, so there is no guarantee of global stability in the general case.

Under some conditions [2], global stability of an SDRE-controlled system can be assured but we need to find a Lyapunov function satisfying the so-called star-convex property. In [3], a technique was presented that uses an upper bound on the estimation of the closed-loop system's matrix, to define a bound on the region of attraction (ROA), and then an estimate of the stability region of the system. The procedure is, however, cumbersome from the computational standpoint for medium-high order systems.

The present paper proposes a less conservative procedure to estimate the ROA of a nonlinear system. The proposed procedure can be applied autonomous as well as to controlled nonlinear systems. The procedure calculates a Lyapunov function V for the linearized system in the neighborhood of the origin (obviously the linearized system must be asymptotically stable), then it is applied to the complete nonlinear system, and the largest level set of V completely inside the region where $\dot{V} < 0$ defines a lower bound of the ROA of the system. If several Lyapunov functions can be defined for the linearized system, the estimation of the ROA is obtained by the union of the different estimates. For a SDRE-controlled system, the method may have limitations, because the analytic SDRE solution is not generally known. The generation of a grid in a region of the state space is necessary, and the procedure must be run on the grid points, resulting in a computationally intensive procedure especially for large size systems.

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II. Review of SDRE Control

The main concepts of SDRE are reviewed next [1]. Consider a nonlinear dynamic system affine in the control:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

and define a quadratic performance index given by

$$J(x,u) = \int_0^\infty [x^T Q(x)x + u^T R(x)u]\,\mathrm{d}t \tag{2}$$

The problem is to find a control law that minimizes (2) subject to the constraints (1). The exact solution of this problem is found by solving the Hamilton–Jacobi–Bellman equation that, unfortunately, is not solvable in analytic form except for very simple cases. To this end, system (1) is parameterized in the state-dependent coefficient (SDC) form, yielding

$$\dot{x} = A(x)x + B(x)u \tag{3}$$

The system and input matrices in (3) are in general functions of the state vector. The control law is found to be

$$u = -K(x)x = -R^{-1}(x)B^T(x)P(x)x \tag{4}$$

With $P(x)$ solution of the state-dependent Riccati equation:

$$A^T(x)P(x) + P(x)A(x) - P(x)B(x)R^1(x)B^T(x)P(x) + Q(x) = 0 \tag{5}$$

The exact solution of (5) is not known in general, and numerical procedures are used to this end [4,5].

The stability property of SDRE-controlled systems can be guaranteed only in a neighborhood of the origin [1]. In some cases the global stability is assured but there is the need of a Lyapunov function satisfying the star-convex property [2].

III. Region of Attraction (ROA)

Because of the above-mentioned stability issues, computation of the region of attraction for a SDRE controller is a very important task. This section presents a new procedure for the estimation of the ROA; the procedure is an alternative to that proposed by Alleyne [3], which is the most recent result to the author's knowledge.

A. Alleyne's Procedure

Alleyne's method is reviewed herein [3]. Consider an autonomous nonlinear system from (1) and parameterized as in (3) (with $u = 0$), and define a vector norm p. A matrix M defines an overvaluing system with respect to norm p if and only if the following inequality is verified for each component:

$$D' p(x) \le M(t,x)p(x) \tag{6}$$

where D' is the right-hand derivative operator. At this point if M is

Hurwitz and irreducible in a selected domain $S \subset R^n$, then the ROA estimated is given by the union of the following sets:

$$\begin{aligned} D_1 &= \{x \in R^n; p^T(x)u_m(M^T) \leq \alpha_1\} \subset S \\ D_\infty &= \{x \in R^n; p(x) \leq \alpha_2 u_m(M)\} \subset S \\ D_c &= \{x \in R^n; p(x) \leq c, c \in R^k_+, -Mc > 0\} \subset S \end{aligned} \tag{7}$$

where $u_m(M)$ is the eigenvector associated with the eigenvalues of M with maximum real part, and α_i are constant values found applying Theorem 1 in [3].

This procedure depends strictly on the chosen norm and can produce no solution if, for instance, the selected domain S is too large. Moreover, the control system designer must find the maximum of n^2 scalar functions that in the general case have to be found numerically. The procedure may not lead to a solution in even simple cases. Consider the following asymptotically stable linear autonomous system:

$$\dot{x} = Ax = \begin{pmatrix} -2 & -1 & -1 & -1 \\ -1 & -2 & -1 & -1 \\ -1 & -1 & -2 & -1 \\ -1 & -1 & -1 & -2 \end{pmatrix} x \tag{8}$$

Considering the norm $p = (|x_1|\ |x_2|\ \cdots\ |x_n|)$ used by Alleyne in his examples in [3], the resulting overvaluing matrix is constant, and thus it is independent from S:

$$M = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix} \tag{9}$$

If M, as in this case, is unstable, the procedure fails.

To estimate the ROA of SDRE-controlled systems the procedure is similar with the only difference being that the overvaluing matrix has to be found with respect to the closed-loop matrix resulting from the SDRE control. This implies that the SDRE has to be solved pointwise because the closed form solution is not known in general; thus gridding of the state space is necessary.

B. Alternative Procedure

As an improvement on ROA estimation, consider the following:

1) Given the autonomous system $\dot{x} = f(x)$ with $x \in \Re^n$. Suppose that the origin is an equilibrium point, then linearize the system in the neighborhood of the origin obtaining

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{0}}$$

If the origin is not an equilibrium point, find a equilibrium point x_{eq} of the system and then translate the state variables such that the new system has the origin as an equilibrium point.

2) If A is Hurwitz, find a Lyapunov function

$$V(x) = x^T P x \tag{10}$$

for this system solving a standard Lyapunov equation

$$A^T P + PA = -Q \tag{11}$$

The only parameter that needs to be set is the value of the positive-definite matrix Q. If the linearized system is unstable, then the equilibrium point will surely be an unstable one; thus the ROA collapses to the equilibrium point. If the linearized system is marginally stable then there is nothing we can say about the nonlinear equilibrium point; anyway, we can choose a simple Lyapunov function candidate, for example, a quadratic function.

3) Apply $V(x)$ to the nonlinear system obtaining

$$\dot{V}(x) = 2x^T P f(x) \tag{12}$$

and find the state-space region L defined by

$$L = \{x \in \Re^n \mid \dot{V}(x) < 0\} \tag{13}$$

4) Find the lowest value of V such that $\dot{V}(x) \geq 0$

$$\bar{V} = \inf\{V(x) \mid x \in \Re^n - L\} \tag{14}$$

and the highest value of V such that the corresponding level set is entirely inside L

$$V_M = \sup\{V(x) \mid V(x) < \bar{V}\} \tag{15}$$

5) The simply connected area E defined by

$$E = \{x \in \Re^n \mid V(x) \leq V_M\} \tag{16}$$

is certainly belonging to the ROA accordingly to the Lyapunov theorem for local stability.

A quick analysis shows that this procedure is always capable of providing an estimate of the ROA. We point out that the correct value of V_M must be found such that all the level sets relative to a value $V' \leq V_M$ must be completely inside L. Equation (14) is thus necessary to bind the maximum level value allowed.

Estimation of the ROA for SDRE-controlled systems requires studying the stability of the closed-loop system

$$\dot{x} = [A(x) - B(x)K(x)]x = A_{\text{CL}}(x)x \tag{17}$$

This is autonomous as in the previous case.

With the proposed method, in the case of autonomous systems it is sufficient to study the scalar function $V(x)$. If the SDRE control matrix $K(x)$ is not available in closed form, the procedure is similar with the only difference being that the SDRE (5) has to be solved pointwise (with an appropriate grid) to find the closed-loop matrix. Independently from the selected grid, the following assumption is necessary: $\dot{V}(x)$ must not change sign along the level set defined by $V(x) = V_M$ and $\dot{V}(x) < 0$; $\forall x \in E$. This implies that the gridding must be chosen fine enough to avoid errors due to large system nonlinearity and nonsmooth functions.

C. Procedures Comparison

Alleyne's procedure seeks the extremum values of n^2 scalar functions, then it computes the eigenvalues and the eigenvectors of the overvaluing matrix M, and finally the union of the three sets in (7) gives the estimated ROA. Our method, instead, considers a single scalar function and then directly produces a ROA estimate. The numerical solution of the SDRE requires the same computational load in both cases, so our procedure is overall less computationally intensive.

Both methods require a grid of the state space. Obviously the finer the grid, the better the results, and the higher the computational burden. We must point out that the proposed procedure is conservative and, under the assumptions, it always produces an estimate. Alleyne's procedure implicitly requires the same hypothesis because of the necessary gridding of the state space, and so the numerical issues of the two methods are the same.

In the following, three examples are shown where gridding of the state space is explicitly done. In the general case the distance between adjacent grid points should be chosen according to the smoothness of the equation describing the system dynamics: $f(x) = A_{\text{CL}}(x)x$. Irregular gridding is possible as well: in fact, a finer grid may be used where the system dynamics is faster and a coarser grid may suffice where it is slower.

The first example is taken from [3], while the last is taken from [6].

D. ROA Estimation for an Autonomous System

Consider the following autonomous system:

$$\dot{x} = A(x)x = \begin{bmatrix} -3 + x_1^2 & x_2 \sin(x_2) \\ 4x_1^2 \cos(x_2) & -3 + x_2^2 \end{bmatrix} x \tag{18}$$

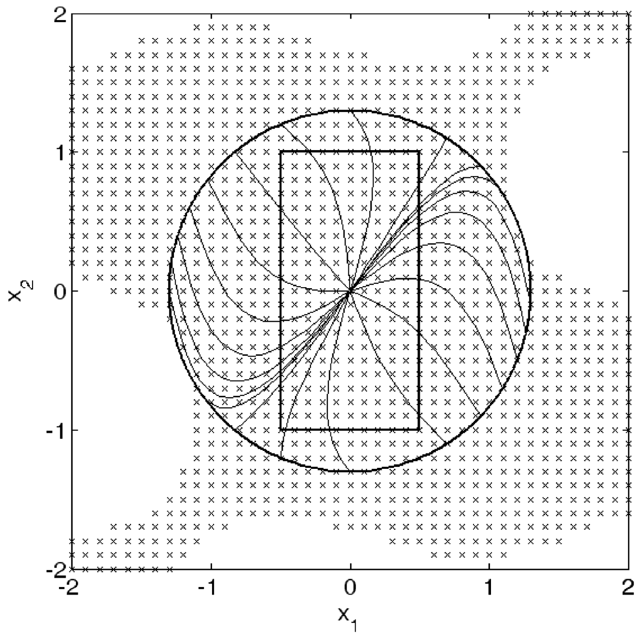

Fig. 1 ROA estimate: circle (new method); rectangle ([3]).

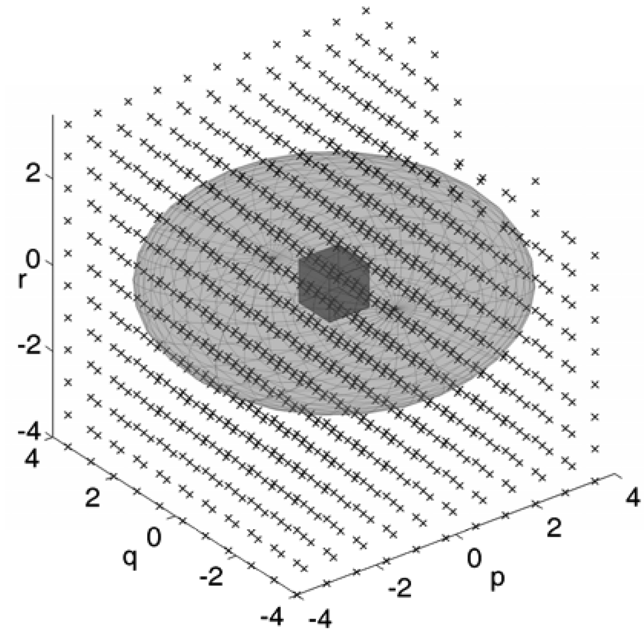

Fig. 2 ROA estimation: cube (from [6]); ellipsoid (new method).

Choose the vector norm:

$$p = \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix} \tag{19}$$

The selected domain S is given by

$$S = \{x_1, x_2 \in R\colon\ |x_1| < 1, |x_2| < 1\} \tag{20}$$

From [3]:

$$M = \begin{bmatrix} \max(-3 + x_1^2) & \max|x_2 \sin(x_2)| \\ \max|4x_1^2 \cos(x_2)| & \max(-3 + x_2^2) \end{bmatrix} = \begin{bmatrix} -2 & 0.842 \\ 4 & -2 \end{bmatrix} \tag{21}$$

The ROA estimation is then given by the union of the following sets:

$$\begin{aligned} D_1 &= \{x \in \Re^2\colon\ |x_1| + 0.459|x_2| < 0.459\} \subset S \\ D_\infty &= \{x \in \Re^2\colon\ |x_1| < 0.459, |x_2| < 1\} \subset S \\ D_c &= \{x \in \Re^2\colon\ |x_1| < 0.5|x_2| < 1\} \subset S \end{aligned} \tag{22}$$

The new procedure for ROA estimation proceeds from the system linearized about the origin, which is diagonal. We can select the Lyapunov function:

$$A(0) = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \Rightarrow V = \frac{1}{2}(x_1^2 + x_2^2) \tag{23}$$

The ROA estimates is a circle of radius 1.3 centered at the origin. The above results are summarized in Fig. 1 where the inner rectangle is the ROA estimated with the Alleyne procedure, and the outer circle is the ROA estimated with the new method; the small crosses represent the grid points where $\dot{V} < 0$. Figure 1 shows some phase-plane trajectories of the controlled system starting from different initial conditions on the edge of the estimated ROA. It is interesting to note that in the neighborhood of the regions where $\dot{V}(x) > 0$ the trajectories have a smaller convergence velocity than the trajectories starting in the other regions.

E. ROA Estimation for a SDRE-Controlled System

Consider the application of the proposed method to the rotational dynamics of flight vehicle [6] described by

$$\begin{cases} \dot{p} = c_1 rq + c_2 pq + c_3 L + c_4 N \\ \dot{q} = c_5 pr - c_6(p^2 - r^2) + c_7 M \\ \dot{r} = c_8 pq - c_2 rq + c_4 L + c_9 M \end{cases} \tag{24}$$

where the state vector $x^T = [p, q, r]$ is the vehicle's angular velocity vector and c_1–c_9 are coefficients that depend on the inertia matrix characteristics. The complete system is described in [6] where the Alleyne method is used to assure the stability of the system in a cubic domain defined by $S = \{p, q, r||p|, |q|, |r| < 0.5\}$. Using the proposed method and choosing

$$Q = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{25}$$

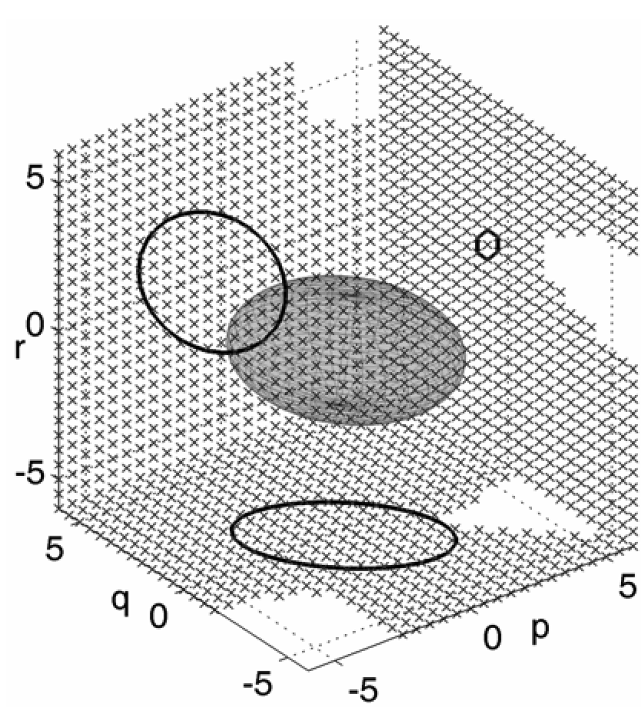

Fig. 3 Projections of the critical level set.

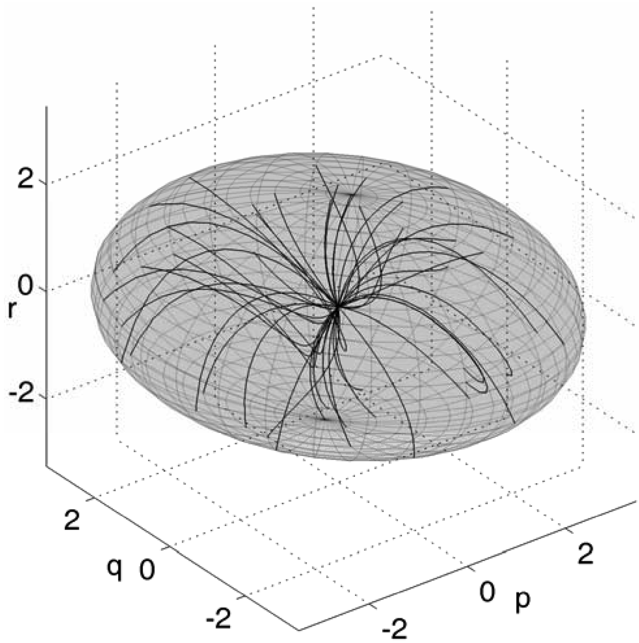

Fig. 4 Simulation results.

yields the Lyapunov function $V = x^T P x$ where

$$P = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix} \tag{26}$$

The procedure gives a larger ROA estimate which is depicted in Fig. 2, where Alleyne's ROA estimation is represented by the cube, as computed in [6], and the V level set containing the ROA is the larger ellipsoid enclosing it.

For a clearer understanding of the results, Fig. 3 shows the sections of the ellipsoid with three planes parallel to the three reference planes, passing through the point relative to the maximum level set of V with $\dot{V} < 0$ in the grid.

Figure 3 shows that the selected level set is the maximum allowed: if we choose a larger level set, the projection on the pq plane would intersect the prohibited zone with $\dot{V} > 0$ yielding a nonadmissible ROA estimation.

Figure 4 shows the results of a series of simulations performed taking as initial conditions points on the surface of the ellipsoid and the resulting trajectories. A refined estimate of the ROA can be found using different Lyapunov functions, that is, setting different values of the known matrix Q in (11); the total ROA estimate is thus given by the union of all the regions of attraction found for all choices of Q.

IV. Conclusions

This paper presented a new method for estimating a higher bound of the region of attraction of a nonlinear system and compared it to current available techniques. The methodology can be applied to SDRE-controlled systems improving the stability properties of such systems. Numerical computation of the proposed procedure can be extensive especially for large systems, however, lower than methods available from the literature.

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